On the blow up phenomenon for the mass critical focusing Hartree equation in $\mathbb{R}^4$

Changxing Miao[†] Guixiang Xu[†] and Lifeng Zhao [‡]
[†]Institute of Applied Physics and Computational Mathematics
P. O. Box 8009, Beijing, China, 100088
[‡] Department of Mathematics, University of Science and Technology of China
(miao_changxing@iapcm.ac.cn, xu_guixiang@iapcm.ac.cn, zhao_lifeng@iapcm.ac.cn)

Abstract

We characterize the dynamics of the finite time blow up solutions with minimal mass for the focusing mass critical Hartree equation with $H^1(\mathbb{R}^4)$ data and $L^2(\mathbb{R}^4)$ data, where we make use of the refined Gagliardo-Nirenberg inequality of convolution type and the profile decomposition. Moreover, we also analyze the mass concentration phenomenon of such blow up solutions.

Key Words: Blow up; Focusing; Hartree equation; Mass critical; Mass concentration; Profile decomposition.
AMS Classification: 35Q40, 35Q55, 47J35.

1 Introduction

In this paper, we consider the Cauchy problem for the following Hartree equation

$$\begin{cases} iu_t + \Delta u = f(u), & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ u(0) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here $f(u) = \lambda\big(V*|u|^2\big)u$, $V(x) = |x|^{-\gamma}, 0 < \gamma < d$, and $*$ denotes the convolution in $\mathbb{R}^d$. If $\lambda > 0$, we call the equation (1.1) defocusing; if $\lambda < 0$, we call it focusing. This equation describes the mean-field limit of many-body quantum systems; see, e.g., [6], [7] and [36]. An essential feature of Hartree equation is that the convolution kernel $V(x)$ still retains the fine structure of micro two-body interactions of the quantum system. By contrast, NLS arise in further limiting regimes where two-body interactions are modeled by a single real parameter in terms of the scattering length. In particular, NLS cannot provide effective models for quantum system with long-range interactions such as the physically important case of the Coulomb potential $V(x) \sim |x|^{-(d-2)}$ in $d \geq 3$, whose scattering length is infinite.

There are many works on the global well-posedness and scattering of equation (1.1). For the defocusing case with $2 < \gamma < \min(4, d)$, J. Ginibre and G. Velo [8] proved the global well-posedness and scattering results in the energy space. Later, K. Nakanishi [30] made use of a new Morawetz estimate to obtain the similar results for the more general functions $V(x)$. Recently, the authors proved the global wellposedness and scattering for the defocusing, energy critical Hartree equation, see [26] and [27]. The global wellposedness and scattering of the focusing, energy critical Hartree equation can refer to [15] and [28]. In this paper, we mainly aim to characterize the dynamics of the finite time blow up solutions with minimal mass for the focusing L^2-critical Hartree equation with $H^1(\mathbb{R}^4)$ data and $L^2(\mathbb{R}^4)$ data.

Now we recall the related results about the focusing mass critical Schrödinger equation

$$iu_t + \Delta u = -|u|^{\frac{4}{d}} u, \quad u(0) = u_0, \tag{1.2}$$

where d is the spatial dimension. Equation (1.2) is called mass critical due to scaling invariance. If $u_0 \in H^1$ is radial, the mass concentration phenomena of the blow up solution was observed near the blow-up time in [20]. Later on, the radial assumption was removed by M. Weinstein [35] and Nawa [31]. For more detailed analysis of the blow up dynamic of (1.2), see [18], [19], [22], [23], [24] and the references therein. If u_0 only lies in L^2, the situation seems quite different because we cannot use the energy conservation law. The pioneering work in this direction is due to J. Bourgain [3] for $d = 2$, where he proved that there exists a blow-up time T^*,

$$\lim_{t \uparrow T^*} \sup_{\substack{\text{cubes } I \subset \mathbb{R}^2, \\ \text{side}(I) < (T^* - t)^{\frac{1}{2}}}} \Big(\int_I |u(t,x)|^2 dx \Big)^{\frac{1}{2}} \geq c(\|u_0\|_{L_x^2}) > 0,$$

where $c(\|u_0\|_{L_x^2})$ is a constant depending on the mass of the initial data. A new proof can be found in S. Keraani [12] by means of the profile decomposition in [21]. Bourgain's result was extended to dimension $d = 1$ by R. Carles and S. Keraani [4] and to dimension $d \geq 3$ by P. Bégout and A. Vargas [2]. Recently, R. Killip, T. Tao and M. Visan [33] established global well-posedness and scattering for (1.2) with radial data in dimension two and mass strictly smaller then that of the ground state. Later R. Killip, M. Visan and X. Zhang [34] extended the results to $d \geq 3$. We dealt with the corresponding problem for the Hartree equation in [29].

This paper is devoted to the study of the blow up behavior of the mass-critical Hartree equation in dimension four:

$$\begin{cases} iu_t + \Delta u = -(|x|^{-2} * |u|^2) u, & \text{in } \mathbb{R}^4 \times \mathbb{R}, \\ u(0) = u_0(x), & \text{in } \mathbb{R}^4. \end{cases} \tag{1.3}$$

The corresponding free equation is

$$\begin{cases} iu_t + \Delta u = 0, & \text{in } \mathbb{R}^4 \times \mathbb{R}, \\ u(0) = u_0(x), & \text{in } \mathbb{R}^4. \end{cases} \tag{1.4}$$

Note that $\gamma = 2$ is the unique exponent which is mass-critical in the sense that the natural scaling

$$u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x),$$

leaves the mass invariant. At the same time, $|x|^{-2}$ is just the physically important case of Coulomb potential for dimension $d = 4$. Moreover, equation (1.3) also possesses the pseudo-conformal symmetry: If $u(t,x)$ solve (1.3), then so does:

$$v(t,x) = \frac{1}{|T-t|^2} \overline{u}\Big(\frac{1}{t-T}, \frac{x}{t-T}\Big) e^{i \frac{|x|^2}{4(t-T)}}. \tag{1.5}$$

We firstly deal with equation (1.3) with data in $H^1(\mathbb{R}^4)$. For the solution $u(t) \in H^1$ of (1.3), there are the following conserved quantities:

$$M(u(t)) = \|u(t)\|_{L_x^2} = \|u(0)\|_{L_x^2},$$

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^4} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^2} dx dy = E(u(0)).$$

According to the local wellposedness theory [5], [25], the solution $u(t) \in H^1(\mathbb{R}^4)$ of (1.3) blows up at finite time T if and only if

$$\lim_{t \to T} \|\nabla u(t)\|_{L^2} \to +\infty.$$

The blow-up theory is mainly connected to the notion of ground state: the unique radial positive solution of the elliptic equation

$$-\Delta Q + Q = (V * |Q|^2)Q. \tag{1.6}$$

The existence of the positive solution is proved by the concentration compactness principle at the beginning of Section 3, which is close related to a refined Gagliardo-Nirenberg inequality of convolution type:

$$\|u\|_{L^V}^4 \le \frac{2}{\|Q\|_{L^2}^2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2, \tag{1.7}$$

where the definition of L^V norm is given by (1.9). The radial symmetry of the positive solution can be obtained from [17]. By adapting Lieb's uniqueness proof in [16] for the ground states $\phi \in H^1$ of the Choquard-Pekar equation ($V(x) = |x|^{-1}$ in dimension $d = 3$), the analogous result for (1.6) can be obtained. See details in [13]. However, the uniqueness proof strongly depends on the specific features of equation (1.6). It is different from the corresponding results for semilinear elliptic equation in [14]. As our result (Theorem 1.1) depends on the uniqueness of the ground state of equation (1.6), it is the reason why we do for the case $d = 4$.

Together with the notion of the ground state Q, the invariance (1.5) yields an explicit blow-up solutions such that $\big\|u\big\|_{L^2} = \big\|Q\big\|_{L^2}$. One can ask if there are other finite time blow up solutions of (1.3) with minimal mass $\big\|Q\big\|_{L^2}$ and how to characterize the dynamics of such blow up solutions near the blow up time.

Now, we can characterize the finite time blow-up solutions with minimal mass in $H^1(\mathbb{R}^4)$.

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{R}^4)$ such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and u be the blow up solution of (1.3) at finite time T, then there exists $x_0 \in \mathbb{R}^4$ such that $e^{i\frac{|x-x_0|^2}{4T}} u_0 \in \mathcal{A}$, where*

$$\mathcal{A} = \Big\{ \rho^2 e^{i\theta} Q(\rho x + y), y \in \mathbb{R}^4, \rho \in \mathbb{R}_*^+, \theta \in [0, 2\pi) \Big\}.$$

Theorem 1.2. *Let u be a solution of (1.3) which blows up at finite time $T > 0$ with initial data $u_0 \in H^1(\mathbb{R}^4)$, and $\lambda(t) > 0$ such that $\lambda(t)\|\nabla u\|_{L^2} \to +\infty$ as $t \uparrow T$. Then there exists $x(t) \in \mathbb{R}^4$ such that*

$$\liminf_{t \uparrow T} \int_{|x - x(t)| \le \lambda(t)} |u(t,x)|^2 dx \ge \int_{\mathbb{R}^4} |Q|^2 dx.$$

The corresponding result of Theorem 1.1 for the Schrödinger equation has been established by F. Merle in [19]. The corresponding result for Theorem 1.2 was proved by M. Weinstein in [35]. T. Hmidi and S. Keraani gave a direct and simplified proof of the above results in [9]. The new ingredient for the Hartree equation is the refined Gagliardo-Nirenberg inequality of the convolution type (1.7), whose proof is based on the well-known concentration compactness method and thus one has to deal with the intertwining of convolution and orthogonality.

Next we consider the blow up behavior of (1.3) with L^2 data. In [25], we showed that for any $u_0 \in L^2(\mathbb{R}^4)$, there exists a unique maximal solution u to (1.3), with

$$u \in C((-T_*, T^*), L^2(\mathbb{R}^4)) \cap L^3_{loc}((-T_*, T^*), L^3(\mathbb{R}^4)),$$

and we have the following alternative: either $T_* = T^* = +\infty$ or

$$\min\{T_*, T^*\} < +\infty \ \text{ and } \ \|u\|_{L_t^3((-T_*,T^*),\ L_x^3)} = +\infty.$$

Moreover, there exists $\delta > 0$ such that if

$$\|u_0\|_{L^2} < \delta, \tag{1.8}$$

the initial value problem (1.3) has a unique global solution $u(t,x) \in L^3_{t,x}(\mathbb{R}\times\mathbb{R}^4)$. We define δ_0 as the supremum of δ in (1.8) such that the global existence for Cauchy problem (1.3) holds, with $u \in (C\cap L^\infty)(\mathbb{R}, L^2(\mathbb{R}^4)) \cap L^3(\mathbb{R}\times\mathbb{R}^4)$. Then in the ball $B_{\delta_0} := \{u_0, \|u_0\|_{L^2} < \delta_0\}$, (1.3) admits a complete scattering theory with respect to the associated linear problem. Similar to the focusing mass-critical Schrödinger equation, we also conjecture that δ_0 should be $\|Q\|_{L^2}$ for the Hartree equation. We have verified the conjecture for radial data in [29]. For general data, it remains open.

Definition 1.1. *Let $u_0 \in L^2(\mathbb{R}^4)$. A solution of (1.3) is said to be a blow-up solution for $t > 0$, if $T^* < +\infty$ or*

$$T^* = +\infty \quad \textit{and} \quad \|u\|_{L_t^3((0,+\infty),\ L_x^3)} = +\infty.$$

Similarly for $t < 0$.

Now we are in position to state the existence of the blow up solutions in both time directions with minimal mass in $L^2(\mathbb{R}^4)$.

Theorem 1.3. *There exists an initial data $u_0 \in L^2(\mathbb{R}^4)$ with $\|u_0\|_{L^2} = \delta_0$, for which the solution of (1.3) blows up for both $t > 0$ and $t < 0$.*

As a direct consequence of the above theorem and the pseudo-conformal transform (1.5), we obtain the existence of the finite time blow up solutions with minimal mass in $L^2(\mathbb{R}^4)$.

Corollary 1.1. *There exists an initial data $u_0 \in L^2(\mathbb{R}^4)$ with $\|u_0\|_{L^2} = \delta_0$, for which the solutions of (1.3) blows up at finite time $T^* > 0$.*

Theorem 1.4. *Let u be a blow up solution of (1.3) at finite time $T^* > 0$ such that $\|u_0\|_{L^2} < \sqrt{2}\delta_0$. Let $\{t_n\}_{n=1}^\infty$ be any time sequence such that $t_n \uparrow T^*$ as $n \to \infty$, and let $\lambda(t) > 0$, such that*

$$\frac{\sqrt{T^*-t}}{\lambda(t)} \to 0, \ \textit{as } t \uparrow T^*.$$

Then there exist a subsequence of $\{t_n\}_{n=1}^\infty$ (still denoted by $\{t_n\}$) and $x(t) \in \mathbb{R}^4$ that satisfy the following properties.

- *(i) There exists a function $\psi \in L^2(\mathbb{R}^4)$ with $\|\psi\|_{L^2} \geq \delta_0$ such that the solution U of (1.3) with initial data ψ blows up for both $t > 0$ and $t < 0$.*
- *(ii) There exists a sequence $\{\rho_n, \xi_n, x_n\}_{n=1}^\infty \subset \mathbb{R}_+^* \times \mathbb{R}^4 \times \mathbb{R}^4$ such that*

  $$\rho_n^2 e^{ix\cdot\xi_n} u(t_n, \rho_n x + x_n) \rightharpoonup \psi, \quad \textit{weakly in} \ \ L^2.$$

  Furthermore, we have

  $$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T^*-t_n}} \leq \frac{1}{\sqrt{T^{**}}}$$

  where T^{**} denotes the lifespan of U.

(iii)

$$\liminf_{t \uparrow T^*} \int_{|x-x(t)| \leq \lambda(t)} |u(x,t)|^2 dx \geq \delta_0^2.$$

Corollary 1.2. *Let u be a blow up solution with minimal mass of (1.3) at finite time $T^* > 0$. Let $\{t_n\}_{n=1}^{\infty}$ be any time sequence such that $t_n \uparrow T^*$ as $n \to \infty$. Then there exists a subsequence of $\{t_n\}_{n=1}^{\infty}$ (still denoted by $\{t_n\}_{n=1}^{\infty}$) and $x(t) \in \mathbb{R}^4$ that satisfy the following properties:*

(i) *There exists a function $\psi \in L^2(\mathbb{R}^4)$ with $\|\psi\|_{L^2} \geq \delta_0$ such that the solution U of (1.3) with initial data ψ blows up for both $t > 0$ and $t < 0$.*

(ii) *There exists a sequence $\{\rho_n, \xi_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^* \times \mathbb{R}^4 \times \mathbb{R}^4$ such that*

$$\rho_n^2 e^{ix\cdot\xi_n} u(t_n, \rho_n x + x_n) \to \psi, \quad \text{strongly in } \ L^2.$$

Furthermore, we have

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq \frac{1}{\sqrt{T^{**}}}$$

where T^{**} denotes the lifespan of U.

(iii)

$$\liminf_{t \uparrow T^*} \int_{|x-x(t)| \leq \lambda(t)} |u(x,t)|^2 dx \geq \delta_0^2.$$

Similar results for the nonlinear Schrödinger equation have appeared in F. Merle, L. Vega [21] and S. Keraani [12]. Since the nonlinearity is non-local for the Hartree equation, we have to pursue suitable decomposition in physical space to exploit the orthogonality.

We will often use the notations $a \lesssim b$ and $a = O(b)$ to mean that there exists some constant C such that $a \leq Cb$. The derivative operator ∇ refers to the derivative with respect to space variable only. We also occasionally use subscripts to denote the spatial derivatives and use the summation convention over repeated indices.

For $1 \leq p \leq \infty$, we define the dual exponent p' by $\frac{1}{p} + \frac{1}{p'} = 1$. For any time interval I, we use $L_t^q L_x^r(I \times \mathbb{R}^4)$ to denote the spacetime Lebesgue norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^4)} := \left(\int_I \|u\|_{L^r(\mathbb{R}^4)}^q dt \right)^{1/q}$$

with the usual modifications when $q = \infty$. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

We say that a pair (q, r) is admissible if

$$\frac{2}{q} = 4\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad 2 \leq q \leq +\infty.$$

For a spacetime slab $I \times \mathbb{R}^4$, we define the *Strichartz* norm $\dot{S}^0(I)$ by

$$\|u\|_{\dot{S}^0(I)} := \sup_{(q,r) \text{ admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^4)}.$$

and define $\dot{S}^1(I)$ by

$$\|u\|_{\dot{S}^1(I)} := \|\nabla u\|_{\dot{S}^0(I)}.$$

We also define $\dot{\mathcal{N}}^0$ as the Banach dual space of $\dot{S}^0$.

Throughout this paper, we denote

$$\|u\|_{L^V} := \Big(\iint |u(x)|^2 V(x-y)|u(y)|^2 dxdy\Big)^{\frac{1}{4}}. \tag{1.9}$$

The rest of this paper is organized as follows: In Section 2, we recall the preliminary estimates such as Strichartz estimates and Virial identity. In Section 3, we prove Theorem 1.1 and Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.3 and Theorem 1.4.

2 Preliminaries

We now recall some useful estimates. First, we have the following *Strichartz* inequalities

Lemma 2.1 ([5], [10]). *Let u be an $\dot{S}^0(I)$ solution to the Schrödinger equation in (1.1). Then*

$$\|u\|_{\dot{S}^0} \lesssim \|u(t_0)\|_{L^2(\mathbb{R}^4)} + \|f(u)\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^4)}$$

for any $t_0 \in I$ and any admissible pairs (q, r). The implicit constant is independent of the choice of interval I.

By definition, it immediately follows that for any function u on $I \times \mathbb{R}^4$,

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{L_{t,x}^3} \lesssim \|u\|_{\dot{S}^0},$$

where all spacetime norms are taken on $I \times \mathbb{R}^4$.

Lemma 2.2. *Let $f(u)(t,x) = \pm u\big(V * |u|^2\big)(t,x)$, where $V(x) = |x|^{-2}$. For any time interval I and $t_0 \in I$, we have*

$$\Big\|\int_{t_0}^t e^{i(t-s)\Delta} f(u)(s,x)ds\Big\|_{\dot{S}^0(I)} \lesssim \|u\|_{L_{t,x}^3}^3.$$

Proof. By Strichartz estimate, Hardy-Littlewood-Sobolev inequality and Hölder inequality, we have

$$\begin{aligned}\Big\|\int_{t_0}^t e^{i(t-s)\Delta} f(u)(s,x)ds\Big\|_{\dot{S}^0(I)} &\lesssim \|f(u)(t,x)\|_{L_t^1 L_x^2} \\ &\lesssim \|V * |u|^2\|_{L_t^{\frac{3}{2}} L_x^6} \|u\|_{L_{t,x}^3} \\ &\lesssim \|u\|_{L_{t,x}^3}^3.\end{aligned}$$

In addition, we have obtained the Virial identity in the proof of the localized Morawetz estimates [26]. Indeed, let $V_0^a(t) = \displaystyle\int a(x)|u(t,x)|^2 dx$, where $a(x)$ is real-valued and u is the solution of (1.1) with $f(u) = -\big(|x|^{-\gamma} * |u|^2\big)u$. Then we get

$$M_0^a(t) =: \partial_t V_0^a(t) = 2\Im \int a_j u_j \overline{u} dx$$

and

$$\begin{aligned}\partial_t M_0^a(t) =& -2\Im \int a_{jj} u_t \overline{u} dx - 4\Im \int a_j \overline{u}_j u_t dx \\ =& -\int \triangle\triangle a |u|^2 dx + 4\Re \int a_{jk} \overline{u_j} u_k dx \\ & -\iint \big(\nabla a(x) - \nabla a(y)\big)\nabla V(x-y)|u(y)|^2|u(x)|^2 dxdy.\end{aligned} \tag{2.1}$$

Lemma 2.3. *If we choose $a(x) = |x|^2$, then we have*

$$\partial_t M_0^a(t) = 8 \int |\nabla u|^2 dx - 2\gamma \iint V(x-y)|u(y)|^2|u(x)|^2 dxdy. \tag{2.2}$$

Lemma 2.4. *If $a(x) = |x|^2$ and $\gamma = 2$, we have*

$$\partial_t^2 V_0^a(t) = 16E(u(0)). \tag{2.3}$$

If $E(u(0)) < 0$, the nonnegative function $V_0^a(t)$ is concave, so the maximal interval of existence is finite. This yields that the solution of (1.3) must blow up in both directions.

3 The blow-up dynamics of the focusing mass critical Hartree equation with H^1 data

Let $V(x) = |x|^{-2}$, we study the minimizing functional

$$J := \min\{J(u) : u \in H^1(\mathbb{R}^4)\}, \quad \text{where} \quad J(u) := \frac{\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2}{\|u\|_{L^V}^4}.$$

First, we have

Lemma 3.1. *If W is the minimizer of $J(u)$, then W satisfies*

$$\Delta W + \alpha(|x|^{-2} * |W|^2)W = \beta W, \quad \textit{where} \quad \alpha = \frac{2J}{\|W\|_{L^2}^2}; \quad \beta = \frac{\|\nabla W\|_{L^2}^2}{\|W\|_{L^2}^2}. \tag{3.1}$$

Remark 3.1. *If W is minimizer of $J(u)$, then $|W|$ is also a minimizer. Hence, we can assume that W is positive. In fact, we have*

$$-|\nabla W| \leq \nabla |W| \leq |\nabla W|$$

in the sense of distribution. In particular, $|W| \in H^1$ and $J(|W|) \leq J(W)$.

Proof of Lemma 3.1. It follows from the fact that W, the minimizing function, is in $H^1(\mathbb{R}^4)$ and satisfies the Euler-Lagrange equation:

$$\frac{d}{d\varepsilon} J(W + \varepsilon v)\Big|_{\varepsilon=0} = 0.$$

Equivalently, we have

$$\begin{aligned}
&\|\nabla W\|_{L^2}^2 \|W\|_{L^V}^4 \int 2\Re(W\bar{v})dx + \|W\|_{L^2}^2 \|W\|_{L^V}^4 \int 2\Re(\nabla W \nabla \bar{v})dx \\
&\quad - \|\nabla W\|_{L^2}^2 \|W\|_{L^2}^2 \Big(\int (V * 2\Re(W\bar{v}))|W|^2 dx + \int (V * |W|^2) 2\Re(W\bar{v})dx \Big) = 0.
\end{aligned}$$

Since

$$\int (V * 2\Re(W\bar{v}))|W|^2 dx = \int (V * |W|^2) 2\Re(W\bar{v})dx,$$

we have

$$\Delta W + \frac{2J}{\|W\|_{L^2}^2}(V * |W|^2)W = \frac{\|\nabla W\|_{L^2}^2}{\|W\|_{L^2}^2} W.$$

Proposition 3.1. *J is attained at a function u with the following properties:*

$$u(x) = aQ(\lambda x + b), \text{ for some } a \in \mathbb{C}^*, \ \lambda > 0, \text{and any } b \in \mathbb{R}^4.$$

where Q satisfies (1.6). Moreover,

$$J = \frac{\|Q\|_{L^2}^2}{2}.$$

We prove this proposition by the following profile decomposition.

Lemma 3.2 (Profile decomposition [9])**.** *For a bounded sequence $\{u_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^4)$, there is a subsequence of $\{u_n\}_{n=1}^{\infty}$ (still denoted by $\{u_n\}$) and a sequence $\{U^{(j)}\}_{j\geq 1}$ in $H^1(\mathbb{R}^4)$ and for any $j \geq 1$, a family (x_n^j) such that*

(i) If $j \neq k$, $|x_n^j - x_n^k| \to \infty$, as $n \to \infty$.

(ii) For every $l \geq 1$,

$$u_n(x) = \sum_{j=1}^{l} U^{(j)}(x - x_n^j) + r_n^l(x). \tag{3.2}$$

Moreover, for any $p \in (2, 4)$,

$$\limsup_{n\to\infty} \|r_n^l\|_{L^p(\mathbb{R}^4)} \to 0 \quad as \quad l \to +\infty. \tag{3.3}$$

(iii)

$$\|u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|U^{(j)}\|_{L^2}^2 + \|r_n^l\|_{L^2}^2 + o_n(1), \tag{3.4}$$

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla U^{(j)}\|_{L^2}^2 + \|\nabla r_n^l\|_{L^2}^2 + o_n(1). \tag{3.5}$$

Proof of Proposition 3.1. Choose a sequence $\{u_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^4)$ such that $J(u_n) \to J$. Suppose $\|u_n\|_{L^2} = 1$ and $\|u_n\|_{L^V} = 1$, then

$$J(u_n) = \int |\nabla u_n|^2 dx \to J.$$

Note that $\{u_n\}_{n=1}^{\infty}$ is bounded in H^1, by Lemma 3.2, we have (3.2)-(3.5). From (3.4) and (3.5), we have

$$\sum_{j=1}^{l} \|U^{(j)}\|_{L^2}^2 \leq 1, \qquad \sum_{j=1}^{l} \|\nabla U^{(j)}\|_{L^2}^2 \leq J. \tag{3.6}$$

Moreover, by Hölder and Young inequalities, we have

$$\|r_n^l\|_{L^V}^4 \leq \|r_n^l\|_{L^{\frac{8}{3}}}^4.$$

From (3.3), $\limsup_{n\to\infty} \|r_n^l\|_{L^{\frac{8}{3}}} \overset{l\to\infty}{\longrightarrow} 0$. It follows that

$$\limsup_{n\to\infty} \|r_n^l\|_{L^V} \overset{l\to\infty}{\longrightarrow} 0.$$

Moreover,

$$\iint \frac{|\sum_{j=1}^{l} U^{(j)}(x-x_n^j)|^2|\sum_{j=1}^{l} U^{(j)}(y-x_n^j)|^2}{|x-y|^2}dxdy$$

$$\leq \sum_{j=1}^{l}\iint \frac{|U^{(j)}(x-x_n^j)|^2|U^{(j)}(y-x_n^j)|^2}{|x-y|^2}dxdy \tag{3.7}$$

$$+\sum_{j=1}^{l}\sum_{k\neq j}\iint \frac{|U^{(j)}(x-x_n^j)||U^{(k)}(x-x_n^k)|(\sum_{i=1}^{l}|U^{(i)}(y-x_n^i)|)^2}{|x-y|^2}dxdy \tag{3.8}$$

$$+\sum_{j=1}^{l}\sum_{k\neq j}\iint \frac{|U^{(j)}(y-x_n^j)||U^{(k)}(y-x_n^k)|(\sum_{i=1}^{l}|U^{(i)}(x-x_n^i)|)^2}{|x-y|^2}dxdy \tag{3.9}$$

$$+\sum_{j=1}^{l}\sum_{k\neq j}\iint \frac{|U^{(j)}(x-x_n^j)|^2|U^{(k)}(y-x_n^k)|^2}{|x-y|^2}dxdy. \tag{3.10}$$

Without loss of generality we can assume that all $U^{(j)}$'s are continuous and compactly supported. Then

$$(3.7)=\sum_{j=1}^{l}\iint \frac{|U^{(j)}(x)|^2|U^{(j)}(y)|^2}{|x-y|^2}dxdy,$$

and by orthogonality, we have

$$(3.8)\leq \sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{k\neq j}\|U^{(i)}(y-x_n^i)\|_{L^{\frac{8}{3}}}^2\|U^{(j)}(\cdot-x_n^j)U^{(k)}(\cdot-x_n^k)\|_{L^{\frac{4}{3}}}\to 0, \quad n\longrightarrow\infty.$$

(3.9) can be similarly estimated. At last, we estimate

$$\begin{aligned}(3.10)&=\sum_{j=1}^{l}\sum_{k\neq j}\iint \frac{|U^{(j)}(x)|^2|U^{(k)}(y)|^2}{|x-y-x_n^j+x_n^k|^2}dxdy\\ &\leq \sum_{j=1}^{l}\sum_{k\neq j}\frac{C}{|x_n^j-x_n^k|^2}\|U^{(j)}\|_{L^2}^2\|U^{(k)}\|_{L^2}^2\to 0, \quad n\longrightarrow\infty.\end{aligned}$$

Therefore, we conclude

$$\Big\|\sum_{j=1}^{l}U^{(j)}(x-x_n^j)\Big\|_{L^V}^4\to\sum_{j=1}^{l}\|U^{(j)}\|_{L^V}^4 \quad \text{as} \quad n\longrightarrow\infty.$$

Thus, we have

$$\lim_{l\to\infty}\sum_{j=1}^{l}\|U^{(j)}\|_{L^V}^4=1.$$

By the definition of J, we have

$$J\|U^j\|_{L^V}^4\leq\|U^{(j)}\|_{L^2}^2\|\nabla U^{(j)}\|_{L^2}^2.$$

So we get that

$$J\sum_{j=1}^{l}\|U^j\|_{L^V}^4\leq\sum_{j=1}^{l}\|U^{(j)}\|_{L^2}^2\|\nabla U^{(j)}\|_{L^2}^2.$$

On the other hand,

$$\sum_{j=1}^{l} \|U^{(j)}\|_{L^2}^2 \|\nabla U^{(j)}\|_{L^2}^2 \leq \sum_{j=1}^{l} \|U^{(j)}\|_{L^2}^2 \sum_{j=1}^{l} \|\nabla U^{(j)}\|_{L^2}^2 \leq J.$$

Thus we conclude that only one term $U^{(j_0)}$ is non-zero, i. e.

$$\|U^{(j_0)}\|_{L^2} = 1; \quad \|U^{(j_0)}\|_{L^V} = 1; \quad \|\nabla U^{(j_0)}\|_{L^2}^2 = J. \tag{3.11}$$

This shows that $U^{(j_0)}$ is the minimizer of $J(u)$. From (3.11), we have

$$\Delta U^{(j_0)} + 2J(|x|^{-2} * |U^{(j_0)}|^2)U^{(j_0)} = JU^{(j_0)}.$$

By Remark 3.1, we can assume that U^{j_0} is positive. Let $U^{(j_0)} = aQ(\lambda x + b)$, where Q is the positive solution of (1.6). An easy computation gives that $\lambda^2 = 2a^2 = J$.

Next we compute the best constant J in terms of Q. Multiplying (1.6) by Q and integrating both sides of this equation, we have

$$-\int |\nabla Q|^2 dx + \int (V * |Q|^2)|Q|^2 dx = \int |Q|^2 dx. \tag{3.12}$$

Since

$$\int (x \cdot \nabla Q) Q dx = -2 \int |Q|^2 dx,$$

$$\int x \cdot \nabla Q \Delta Q dx = -\sum_{i,j} \int \big(\delta_{ij}\partial_i Q \partial_j Q + x_i \partial_i \partial_j Q \partial_j Q\big) = \|\nabla Q\|_{L^2}^2,$$

and

$$\begin{aligned}
\int x \cdot \nabla Q (V * |Q|^2) Q dx &= \frac{1}{2} \int x \cdot \nabla Q^2 (V * |Q|^2) dx \\
&= \frac{1}{2} \int x \cdot \nabla \big((V * |Q|^2) Q^2\big) dx - \frac{1}{2} \int x \cdot (\nabla V * Q^2) Q^2 dx \\
&= -2 \int (V * |Q|^2) Q^2 dx + \iint \frac{x \cdot (x-y)}{|x-y|^4} Q^2(x) Q^2(y) dx dy \\
&= -\frac{3}{2} \|Q\|_{L^V}^4,
\end{aligned}$$

we have

$$\|\nabla Q\|_{L^2}^2 - \frac{3}{2}\|Q\|_{L^V}^4 = -2\|Q\|_{L^2}^2.$$

Together with (3.12), this yields

$$\|\nabla Q\|_{L^2}^2 = \|Q\|_{L^2}^2.$$

So,

$$J = \|\nabla U^{(j_0)}\|_{L^2}^2 = \frac{\|Q\|_{L^2}^2}{2}.$$

So far, we have obtained the existence of the positive solution of (1.6). In addition, Theorem 3 in [13] together with Theorem 1.2 in [17] implies that this positive solution is also radial and unique in $H^1(\mathbb{R}^4)$. Note that the uniqueness proof strongly depends on the specific features of equation (1.6). In fact, the uniqueness of the ground state Q of (1.6) has not be resolved completely for the general potential $V(x)$, and be stated as an open problem in [6].

We first make use of the ground state Q to give a sufficient condition for the global existence of (1.3), which together with (1.5) implies that $\big\|Q\big\|_{L^2}$ is the minimal mass of the blow up solutions.

Theorem 3.1. *If $u_0 \in H^1(\mathbb{R}^4)$ and $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution $u(t)$ of (1.3) is global in time.*

Proof. By the local wellposedness theory, it suffices to prove that for every $t \in \mathbb{R}$, we have

$$\|\nabla u(t)\|_{L^2} < +\infty.$$

Now from Proposition 3.1 and the conservation of mass, we have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\int |\nabla u(t)|^2 dx - \frac{1}{4}\int (V * |u(t)|^2)|u(t)|^2 dx \\ &\geq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{1}{4}\frac{2}{\|Q\|_{L^2}^2}\|u(t)\|_{L^2}^2\|\nabla u(t)\|_{L^2}^2 \\ &= \frac{1}{2}\|\nabla u(t)\|_{L^2}^2\Big(1 - \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2}\Big). \end{aligned} \tag{3.13}$$

Since $\|u_0\|_{L^2} < \|Q\|_{L^2}$, so we have the uniform bound of $\|\nabla u(t)\|_{L^2}^2$. This proves the global existence.

Before we prove Theorem 1.1, we state a proposition in two equivalent forms.

Proposition 3.2 (Static version). *If $u \in H^1(\mathbb{R}^4)$ such that $\|u\|_{L^2} = \|Q\|_{L^2}$ and $E(u) = 0$, then $u(x)$ is of the following form*

$$u(x) = e^{i\theta}\lambda^2 Q(\lambda x + b), \quad \textit{for some} \quad \theta \in \mathbb{R},\ \lambda > 0,\ b \in \mathbb{R}^4.$$

Proof. Since $E(u) = 0$, we have $\|\nabla u\|_{L^2}^2 = \frac{1}{2}\|u\|_{L^V}^4$. So we get

$$J(u) = \frac{\|Q\|_{L^2}^2\|\nabla u\|_{L^2}^2}{\|u\|_{L^V}^4} = \frac{1}{2}\|Q\|_{L^2}^2 = J.$$

By Proposition 3.1 and the uniqueness of the ground state Q, u is of the form $u(x) = aQ(\lambda x + b)$. The condition $\|u\|_{L^2} = \|Q\|_{L^2}$ ensures that $|a| = \lambda^2$. So $u(x) = e^{i\theta}\lambda^2 Q(\lambda x + b)$.

Proposition 3.3 (Dynamic version). *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $H^1(\mathbb{R}^4)$ such that $\|u_n\|_{L^2} = \|Q\|_{L^2}$, $E(u_n) \leq M$ and $\|\nabla u_n\|_{L^2} \to \infty$. We define*

$$\lambda_n := \frac{\|\nabla u_n\|_{L^2}}{\|\nabla Q\|_{L^2}},$$

then there exists a subsequence (still denoted by $\{u_n\}$), a sequence $(y_n) \subset \mathbb{R}^4$ and a real number θ such that

$$e^{i\theta}\lambda_n^{-2}u_n(\lambda_n^{-1}x + y_n) \to Q(x) \ \textit{strongly in}\ H^1. \tag{3.14}$$

Proof. Let

$$\tilde{u}_n(x) = \frac{1}{\lambda_n^2}u_n(\frac{x}{\lambda_n}),$$

then $\|\tilde{u}_n\|_{L^2} = \|Q\|_{L^2}$ and $\|\nabla\tilde{u}_n\|_{L^2} = \|\nabla Q\|_{L^2}$. Moreover,

$$E(\tilde{u}_n) = \frac{E(u_n)}{\lambda_n^2} \to 0, \quad \text{as} \quad n \to \infty.$$

So we have

$$J(\tilde{u}_n) = \|Q\|_{L^2}^2\frac{\|\nabla\tilde{u}_n\|_{L^2}^2}{\|\tilde{u}_n\|_{L^V}^4} = \|Q\|_{L^2}^2\frac{\|\nabla\tilde{u}_n\|_{L^2}^2}{2\|\nabla\tilde{u}_n\|_{L^2}^2 - 4E(\tilde{u}_n)} \quad \longrightarrow \quad \frac{\|Q\|_{L^2}^2}{2} = J, \quad n \longrightarrow \infty.$$

Therefore, by Lemma 3.2, we can choose a subsequence $\tilde{u}_n$ and $(x_n) \subset \mathbb{R}^4$ such that $\tilde{u}_n(x + x_n) \to aQ(\lambda x + b)$ in H^1. The conditions $\|\tilde{u}_n\|_{L^2} = \|Q\|_{L^2}$ and $\|\nabla \tilde{u}_n\|_{L^2} = \|\nabla Q\|_{L^2}$ imply $|a| = \lambda = 1$, so we have (3.14) for $y_n = \lambda_n^{-1}(x_n - b)$.

In order to prove Theorem 1.1, we also need the following lemma. The proof relies heavily on the techniques in V. Banica [1].

Lemma 3.3. *Suppose $u \in H^1(\mathbb{R}^4)$, $\|u\|_{L^2} = \|Q\|_{L^2}$, then for all real function $w \in C^1$ with ∇w is bounded, we have*

$$\Big| \int_{\mathbb{R}^4} \nabla w(x) \Im(u \nabla u)(x) dx \Big| \leq \sqrt{2} E(u)^{\frac{1}{2}} \Big(\int |u|^2 |\nabla w|^2 dx \Big)^{\frac{1}{2}}.$$

Proof. Since

$$\|ue^{isw(x)}\|_{L^2} = \|u\|_{L^2} = \|Q\|_{L^2},$$

for any $s \in \mathbb{R}$, by (3.13) we know that $E(ue^{isw(x)}) \geq 0$. So, for any s,

$$\frac{1}{2} \int_{\mathbb{R}^4} |\nabla u + isu \nabla w|^2 dx - \frac{1}{4} \int_{\mathbb{R}^4} (V * |u|^2)|u|^2 dx \geq 0.$$

Namely,

$$E(u) + s \int_{\mathbb{R}^4} \nabla w \Im(u \nabla u) dx + \frac{s^2}{2} \int_{\mathbb{R}^4} |u|^2 |\nabla w|^2 dx \geq 0.$$

Note that this holds for any s, so the discriminant is non-positive. So we get the result.

Now we turn to the proof of Theorem 1.1 and Theorem 1.2, which is borrowed from [9].

Proof of Theorem 1.1. Suppose $u(t, x)$ is the solution of (1.3) which blows up at T and let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary sequence such that $t_n \uparrow T$. Let $u_n = u(t_n)$, by Proposition 3.3, we have

$$e^{i\theta} \lambda_n^{-2} u_n(\lambda_n^{-1} x + y_n) \to Q(x) \text{ strongly in } H^1.$$

From this we get

$$|u(t_n, x)|^2 dx - \|Q\|_{L^2}^2 \delta_{x = y_n} \rightharpoonup 0. \tag{3.15}$$

where $y_n \to 0$ (up to translation) or $y_n \to \infty$.

Now let $\phi \in C_0^\infty(\mathbb{R}^4)$ be a nonnegative radial function such that

$$\phi(x) = |x|^2, \text{ if } |x| < 1 \text{ and } |\nabla \phi|^2 \leq C\phi(x).$$

For every $p \in \mathbb{N}^*$ we define

$$\phi_p(x) = p^2 \phi(\frac{x}{p}) \text{ and } g_p(t) = \int \phi_p(x) |u(t, x)|^2 dx.$$

By Lemma 3.3, for every $t \in [0, T)$, we have

$$\begin{aligned} |\dot{g}_p(t)| = 2\Big| \int_{\mathbb{R}^4} \nabla \phi_p(x) \Im(u \nabla u)(x) dx \Big| &\leq 2\sqrt{2} E(u_0)^{\frac{1}{2}} \Big(\int |u|^2 |\nabla \phi_p(x)|^2 dx \Big)^{\frac{1}{2}} \\ &\leq CE(u_0)^{\frac{1}{2}} \Big(\int |u|^2 \phi_p(x) dx \Big)^{\frac{1}{2}} \leq C(u_0) \sqrt{g_p(t)}. \end{aligned}$$

Integrating with respect to t, we get that

$$\Big| \sqrt{g_p(t)} - \sqrt{g_p(t_n)} \Big| \leq C(u_0) |t_n - t|.$$

If $y_n \to 0$, then $g_p(t_n) \to \|Q\|_{L^2}^2 \phi_p(0) = 0$ by (3.15); if $|y_n| \to \infty$, also $g_p(t_n) \to 0$ since ϕ_p is compactly supported. So, if we let n go to infinity, we have

$$g_p(t) \leq C(u_0)(T-t)^2.$$

Now fix $t \in [0,T)$ and let p go to infinity, then by (2.3) we get

$$8t^2 E(e^{i\frac{|x|^2}{4t}} u_0) = \int |x|^2 |u(t,x)|^2 dx \leq C(u_0)(T-t)^2. \tag{3.16}$$

Hence

$$|y_n|^2 \|Q\|_{L^2}^2 \leq C(u_0) T^2.$$

Thus y_n can not go to infinity. This implies that $\{y_n\}$ converges to 0. Let t goes to T, from (3.16), we get

$$E(e^{i\frac{|x|^2}{4T}} u_0) = 0.$$

Note also that

$$\|e^{i\frac{|x|^2}{4T}} u_0\|_{L^2} = \|Q\|_{L^2}.$$

By Proposition 3.2, we conclude that $e^{i\frac{|x|^2}{4T}} u_0 \in \mathcal{A}$.

Proof of Theorem 1.2. We denote

$$\rho(t) = \frac{\|\nabla Q\|_{L^2}}{\|\nabla u\|_{L^2}} \text{ and } v(t,x) = \rho^2 u(t, \rho x).$$

Let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary time sequence such that $t_n \uparrow T$, $v_n(x) = v(t_n, x)$, then by mass conservation and the definition of $\rho(t)$, we have

$$\|v_n\|_{L^2} = \|u_0\|_{L^2} \text{ and } \|\nabla v_n\|_{L^2} = \|\nabla Q\|_{L^2}.$$

Since u blows up at time T, we have

$$\rho(t_n) \to 0, \quad \text{as } t_n \to T.$$

So we have

$$E(v_n) = \rho_n^2 E(u_0) \to 0, \text{as } n \to \infty.$$

In particular,

$$\|v_n\|_{L^V}^4 \to 2\|\nabla Q\|_{L^2}^2, \text{as } n \to \infty.$$

According to Lemma 3.2, the sequence $\{v_n\}_{n=1}^{\infty}$ can be written, up to a subsequence, as

$$v_n(x) = \sum_{j=1}^{l} U^{(j)}(x - x_n^j) + r_n^l(x)$$

such that (3.3), (3.4) and (3.5) hold. This implies, in particular, that

$$2\|\nabla Q\|_{L^2}^2 \leq \limsup_{n\to\infty} \|v_n\|_{L^V}^4 = \limsup_{n\to\infty} \Big\| \sum_{j=1}^{\infty} U^j(\cdot - x_n^j) \Big\|_{L^V}^4.$$

As in the discussion of the proof of Proposition 3.1, the pairwise orthogonality of the family $\{x^j\}_{j=1}^{\infty}$, together with (1.6) and (3.5), gives

$$
\begin{aligned}
2\|\nabla Q\|_{L^2}^2 \leq &\sum_{j=1}^{\infty} \|U^j\|_{L^V}^4 \leq \sum_{j=1}^{\infty} \frac{2}{\|Q\|_{L^2}^2} \|U^j\|_{L^2}^2 \|\nabla U^j\|_{L^2}^2 \\
\leq &\frac{2}{\|Q\|_{L^2}^2} \sup_{j\geq 1} \|U^j\|_{L^2}^2 \sum_{j=1}^{\infty} \|\nabla U^j\|_{L^2}^2 \leq \frac{2}{\|Q\|_{L^2}^2} \|\nabla v_n\|_{L^2}^2 \sup_{j\geq 1} \|U^j\|_{L^2}^2 \\
\leq &\frac{2}{\|Q\|_{L^2}^2} \|\nabla Q\|_{L^2}^2 \sup_{j\geq 1} \|U^j\|_{L^2}^2.
\end{aligned}
$$

Therefore, we get that

$$\sup_{j\geq 1} \|U^j\|_{L^2}^2 \geq \|Q\|_{L^2}^2.$$

Since $\sum \|U^j\|_{L^2}^2$ converges, the supremum above is attained. In particular, there exists j_0 such that

$$\|U^{j_0}\|_{L^2}^2 \geq \|Q\|_{L^2}^2.$$

On the other hand, a change of variables gives

$$v_n(x + x_n^{j_0}) = U^{j_0}(x) + \sum_{\substack{1\leq j\leq l \\ j\neq j_0}} U^j(x + x_n^{j_0} - x_n^j) + \tilde{r}_n^l(x),$$

where $\tilde{r}_n^l(x) = r_n^l(x + x_n^{j_0})$. The pairwise orthogonality of the family $\{x^j\}_{j=1}^{\infty}$ implies

$$U^j(\cdot + x_n^{j_0} - x_n^j) \rightharpoonup 0, \text{ weakly}$$

for every $j \neq j_0$. Hence we get

$$r_n(\cdot + x_n^{j_0}) \rightharpoonup U^{j_0} + \tilde{r}^l,$$

where $\tilde{r}^l$ denote the weak limit of $\{\tilde{r}_n^l\}_{n=1}^{\infty}$. However, we have

$$\|\tilde{r}^l\|_{L^V} \leq \limsup_{n\to\infty} \|\tilde{r}_n^l\|_{L^V} = \limsup_{n\to\infty} \|r_n^l\|_{L^V} \overset{l\to\infty}{\longrightarrow} 0.$$

By uniqueness of weak limit, we get

$$\tilde{r}^l = 0$$

for every $l \neq j_0$ so that

$$r_n(\cdot + x_n^{j_0}) \rightharpoonup U^{j_0}, \text{ in } H^1,$$

namely,

$$\rho_n^2 u(t_n, \rho_n \cdot + x_n^{j_0}) \rightharpoonup U^{j_0} \in H^1 \text{ weakly}.$$

Thus for every $A > 0$,

$$\liminf_{n\to+\infty} \int_{|x|\leq A} \rho_n^4 |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x|\leq A} |U^{j_0}|^2 dx.$$

In view of the assumption $\lambda(t_n)/\rho_n \to \infty$, this gives immediately

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x|\leq A} |U^{j_0}|^2 dx$$

for every $A > 0$, which means that

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\le\lambda(t_n)} |u(t_n,x)|^2 dx \ge \int |U^{j_0}|^2 dx \ge \int |Q|^2 dx.$$

Since the sequence $\{t_n\}_{n=1}^{\infty}$ is arbitrary, we infer

$$\liminf_{t\to T} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\le\lambda(t)} |u(t,x)|^2 dx \ge \int |Q|^2 dx.$$

But for every $t \in [0,T)$, the function $y \mapsto \int_{|x-y|\le\lambda(t)} |u(t,x)|^2 dx$ is continuous and goes to 0 at infinity. As a result, we get

$$\sup_{y\in\mathbb{R}^4} \int_{|x-y|\le\lambda(t)} |u(t,x)|^2 dx = \int_{|x-x(t)|\le\lambda(t)} |u(t,x)|^2 dx,$$

for some $x(t) \in \mathbb{R}^4$ and Theorem 1.2 is proved.

4 The blow-up dynamics of the focusing mass critical Hartree equation with L^2 data

In this section we prove Theorem 1.3 and Theorem 1.4.

Definition 4.1. *For every sequence $\mathbf{\Gamma}_n = \{\rho_n, t_n, \xi_n, x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$, we define the isometric operator $\mathbf{\Gamma}_n$ on $L^3_{t,x}(\mathbb{R}\times\mathbb{R}^4)$ by*

$$\mathbf{\Gamma}_n(f)(t,x) = \rho_n^2 e^{ix\cdot\xi_n} e^{-it|\xi_n|^2} f(\rho_n^2 t + t_n, \rho_n(x - t\xi_n) + x_n).$$

Two sequences $\mathbf{\Gamma}^j = \{\rho_n^j, t_n^j, \xi_n^j, x_n^j\}_{n=1}^{\infty}$ and $\mathbf{\Gamma}^k = \{\rho_n^k, t_n^k, \xi_n^k, x_n^k\}_{n=1}^{\infty}$ are said to be orthogonal if

$$\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} \to +\infty$$

or

$$\rho_n^j = \rho_n^k \ \text{ and } \ \frac{|\xi_n^j - \xi_n^k|}{\rho_n^j} + |t_n^j - t_n^k| + \left| \frac{\xi_n^j - \xi_n^k}{\rho_n^j} t_n^j + x_n^j - x_n^k \right| \to +\infty.$$

Lemma 4.1 (Linear profile decomposition [2])**.** *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^4)$. Then there exists a subsequence of $\{\varphi_n\}_{n=1}^{\infty}$ (still denoted by $\{\varphi_n\}_{n=1}^{\infty}$) which satisfies the following properties: there exists a family $\{V^j\}_{j=1}^{\infty}$ of solutions of (1.4) and a family of pairwise orthogonal sequences $\mathbf{\Gamma}^j = \{\rho_n^j, t_n^j, \xi_n^j, x_n^j\}_{n=1}^{\infty}$, such that for every $(t,x) \in \mathbb{R}\times\mathbb{R}^4$, we have*

$$e^{it\Delta}\varphi_n(x) = \sum_{j=1}^{l} \mathbf{\Gamma}_n^j V^j(t,x) + w_n^l(t,x), \tag{4.1}$$

with

$$\limsup_{n\to\infty} \|w_n^l\|_{L^3(\mathbb{R}\times\mathbb{R}^4)} \to 0, \ \text{ as } l \to \infty. \tag{4.2}$$

Moreover, for every $l \ge 1$,

$$\|\varphi_n\|_{L^2}^2 = \sum_{j=1}^{l} \|V^j\|_{L^2}^2 + \|w_n^l\|_{L^2}^2 + o_n(1). \tag{4.3}$$

Definition 4.2. *Let $\Gamma_n = \{\rho_n, t_n, \xi_n, x_n\}_{n=1}^\infty$ be a sequence of $\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$ such that the quantity $\{t_n\}_{n=1}^\infty$ has a limit in $[-\infty, +\infty]$ when n goes to the infinity. Let V be a solution of linear Schrödinger equation (1.4). We say that U is the nonlinear profile associated to $\{V, \Gamma_n\}_{n=1}^\infty$ if U is the unique maximal solution of the nonlinear Schrödinger equation (1.3) satisfying*

$$\Big\|(U-V)(t_n,\cdot)\Big\|_{L^2(\mathbb{R}^4)} \to 0, \quad \text{as } n \to \infty.$$

In order to prove Theorem 1.3 and Theorem 1.4, we first state a key theorem, which is similar to that in [11] and [12] and its proof is the same essence with that of stability theory.

Theorem 4.1 (Nonlinear profile decomposition)**.** *Let $\{\varphi_n\}_{n=1}^\infty$ be a bounded family of $L^2(\mathbb{R}^4)$ and $\{u_n\}_{n=1}^\infty$ the corresponding family of solutions to (1.3) with initial data $\{\varphi_n\}_{n=1}^\infty$. Let $\{V^j, \mathbf{\Gamma}_n^j\}_{j=1}^\infty$ be the family of linear profiles associated to $\{\varphi_n\}_{j=1}^\infty$ via Lemma 4.1 and $\{U^j\}_{j=1}^\infty$ the family of nonlinear profiles associated to $\{V^j, \mathbf{\Gamma}_n^j\}_{j=1}^\infty$ via Definition 4.2. Let $\{I_n\}_{n=1}^\infty$ be a family of intervals containing the origin 0. Then the following statements are equivalent:*
(i) For every $j \geq 1$, we have

$$\lim_{n\to\infty} \|\mathbf{\Gamma}_n^j U^j\|_{L^3_{t,x}[I_n]} < \infty,$$

(ii)

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}[I_n]} < \infty.$$

Moreover, if (i) or (ii) holds, then

$$u_n = \sum_{j=1}^{l} \mathbf{\Gamma}_n^j U^j + w_n^l + r_n^l, \tag{4.4}$$

where w_n^l is as in (4.2) and

$$\lim_{n\to\infty} (\|r_n^l\|_{L^3_{t,x}[I_n]} + \sup_{t\in I_n} \|r_n^l\|_{L^2}) \to 0 \quad \text{as} \quad l \to \infty \tag{4.5}$$

Proof. **Step 1:** We prove (4.4) and (4.5) provided that (i) or (ii) holds. Let

$$r_n^l = u_n - \sum_{j=1}^{l} U_n^j - w_n^l, \text{ where } U_n^j := \mathbf{\Gamma}_n^j U^j,$$

and let $V_n^j := \mathbf{\Gamma}_n^j V^j$, then r_n^l satisfies the following equation

$$\begin{cases} i\partial_t r_n^l + \Delta r_n^l = f_n^l, \\ \qquad r_n^l(0) = \displaystyle\sum_{j=1}^{l} (V_n^j - U_n^j)(0,x). \end{cases} \tag{4.6}$$

where

$$f_n^l := p(W_n^l + w_n^l + r_n^l) - \sum_{j=1}^{l} p(U_n^j),$$

and

$$p(z) := -(|x|^{-2} * |z|^2)z, \quad W_n^l := \sum_{j=1}^{l} U_n^j.$$

It suffices to prove that

$$\lim_{n\to\infty}(\|r_n^l\|_{L^3_{t,x}[I_n]}+\sup_{t\in I_n}\|r_n^l\|_{L^2})\overset{l\to\infty}{\longrightarrow}0. \tag{4.7}$$

By Strichartz estimates and Young's inequality, we have

$$\begin{aligned}
\|r_n^l\|_{L^3_{t,x}[I_n]}+\sup_{t\in I_n}\|r_n^l\|_{L^2}\lesssim&\quad\Big\|p(W_n^l+w_n^l+r_n^l)-\sum_{j=1}^l p(U_n^j)\Big\|_{\dot{\mathcal{N}}^0[I_n]}+\|r_n^l(0,\cdot)\|_{L^2}\\
\lesssim&\quad\Big\|p(W_n^l)-\sum_{j=1}^l p(U_n^j)\Big\|_{\dot{\mathcal{N}}^0[I_n]} &(4.8)\\
&+\Big\|p(W_n^l+w_n^l)-p(W_n^l)\Big\|_{L^1_tL^2_x[I_n]} &(4.9)\\
&+\Big\|p(W_n^l+w_n^l+r_n^l)-p(W_n^l+w_n^l)\Big\|_{L^1_tL^2_x[I_n]} &(4.10)\\
&+\|r_n^l(0,\cdot)\|_{L^2}.
\end{aligned}$$

We will estimate these three terms, respectively. Firstly, we estimate (4.8).

$$\begin{aligned}
(4.8)\leq&\sum_{j_1=1}^l\sum_{j_2\neq j_1}\Big\|(|x|^{-2}*|U_n^{j_1}|^2)U_n^{j_2}\Big\|_{L^{\frac32}_{t,x}[I_n]} &(4.11)\\
&+\sum_{j_1=1}^l\sum_{j_2\neq j_1}\sum_{j_3=1}^l\Big\|\big(|x|^{-2}*(U_n^{j_1}U_n^{j_2})\big)U_n^{j_3}\Big\|_{L^1_tL^2_x[I_n]}. &(4.12)
\end{aligned}$$

Without loss of generality we can assume that both U^{j_1} and U^{j_2} have compact support in t and x. Let $V(x)=|x|^{-2}$, then we have

$$\begin{aligned}
&\iint|(V*|U_n^{j_1}|^2)U_n^{j_2}|^{\frac32}dxdt\\
=&\iint\Big|\int(\rho_n^{j_1})^4|U^{j_1}((\rho_n^{j_1})^2t+t_n^{j_1},\rho_n^{j_1}(x-y-t\xi_n^{j_1})+x_n^{j_1})|^2V(y)dy\\
&\qquad\times(\rho_n^{j_2})^2U^{j_2}((\rho_n^{j_2})^2t+t_n^{j_2},\rho_n^{j_2}(x-t\xi_n^{j_2})+x_n^{j_2})\Big|^{\frac32}dxdt\\
=&\Big(\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\Big)^3\iint\Big|\int|U^{j_1}(\tilde t,\tilde x-\tilde y)|^2V(\tilde y)d\tilde yU^{j_2}\Big(\Big(\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\Big)^2\tilde t-\Big(\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\Big)^2t_n^{j_1}+t_n^{j_2},\\
&\qquad\frac{\rho_n^{j_2}}{\rho_n^{j_1}}\tilde x+\frac{\rho_n^{j_2}(\xi_n^1-\xi_n^2)}{(\rho_n^{j_1})^2}\tilde t-\frac{\rho_n^{j_2}(\xi_n^{j_1}-\xi_n^{j_2})}{(\rho_n^{j_1})^2}t_n^{j_1}-\frac{\rho_n^{j_2}x_n^{j_1}}{\rho_n^{j_1}}+x_n^{j_2}\Big)\Big|^{\frac32}d\tilde xd\tilde t.
\end{aligned}$$

If $\rho_n^{j_2}/\rho_n^{j_1}+\rho_n^{j_1}/\rho_n^{j_2}\to+\infty$ or $|t_n^{j_1}-t_n^{j_2}|\to+\infty$, by the compact support assumption on t, we conclude that (4.11) $\to 0$. Otherwise, by orthogonality we have

$$\frac{|\xi_n^{j_1}-\xi_n^{j_2}|}{\rho_n^{j_1}}+\Big|\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}t_n^{j_1}+x_n^{j_1}-x_n^{j_2}\Big|\to+\infty. \tag{4.13}$$

Without loss of generality, we may assume that $\rho_n^{j_2}/\rho_n^{j_1}\to 1$. Then the complicated expression of the function U^{j_2} of $\tilde t$ and $\tilde x$ can be simplified to

$$U^{j_2}\Big(\tilde t-t_n^{j_1}+t_n^{j_2},\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}\tilde t+\tilde x-x_n^{j_1}+x_n^{j_2}-\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}t_n^{j_1}\Big).$$

Meanwhile, we have

$$\int |U^{j_1}(\tilde{t},\tilde{x}-\tilde{y})|^2 V(\tilde{y})d\tilde{y} \le \int_{|\tilde{y}|\le 1} |U^{j_1}(\tilde{t},\tilde{x}-\tilde{y})|^2 V(\tilde{y})d\tilde{y} + \sum_{j=0}^{\infty} \int_{2^j\le|\tilde{y}|\le 2^{j+1}} |U^{j_1}(\tilde{t},\tilde{x}-\tilde{y})|^2 V(\tilde{y})d\tilde{y}.$$

Note that U^{j_1} is compactly supported in x, so for any fixed j,

$$\int_{2^j\le|\tilde{y}|\le 2^{j+1}} |U^{j_1}(\tilde{t},\cdot-\tilde{y})|^2 V(\tilde{y})d\tilde{y}$$

is also compactly supported. Thus (4.13) implies that for any $j_1 \neq j_2$,

$$\lim_{n\to\infty} \iint \left| \int_{2^j\le|\tilde{y}|\le 2^{j+1}} |U^{j_1}(\tilde{t},\cdot-\tilde{y})|^2 V(\tilde{y})d\tilde{y} U^{j_2}\bigg(\tilde{t}-t_n^{j_1}+t_n^{j_2}, \right.$$
$$\left. \frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}\tilde{t}+\tilde{x}-x_n^{j_1}+x_n^{j_2}-\frac{\xi_n^{j_1}-\xi_n^{j_2}}{\rho_n^{j_1}}t_n^{j_1}\bigg)\right|^{\frac{3}{2}} d\tilde{x}d\tilde{t} = 0.$$

Therefore, we get that (4.11) $\to 0$ as $n\to\infty$.

On the other hand,

$$\left\|(|x|^{-2} * (U_n^{j_1}U_n^{j_2})U_n^{j_3}\right\|_{L_t^1L_x^2[I_n]} \le C\left\|U_n^{j_1}U_n^{j_2}\right\|_{L_{t,x}^{\frac{3}{2}}}\left\|U_n^{j_3}\right\|_{L_{t,x}^3}.$$

By orthogonality,

$$\left\|U_n^{j_1}U_n^{j_2}\right\|_{L_{t,x}^{\frac{3}{2}}} \to 0, \quad \text{as} \quad n\to\infty.$$

Because $\left\|U_n^{j_3}\right\|_{L_{t,x}^3}$ is bounded, we have

$$(4.12) \xrightarrow{n\to\infty} 0.$$

Next, we prove that

$$\lim_{l\to\infty}\left(\lim_{n\to\infty}\|W_n^l+w_n^l\|_{L_{t,x}^3[I_n]}\right) \le C.$$

From (4.3), we have

$$\|w_n^l\|_{L_{t,x}^3[I_n]} \le C\|w_n^l(0)\|_{L^2} \le C\|\varphi_n\|_{L^2}.$$

It suffices to verify

$$\lim_{l\to\infty}\left(\lim_{n\to\infty}\|W_n^l\|_{L_{t,x}^3[I_n]}\right) \le C. \tag{4.14}$$

From the orthogonality of Γ_n^j, as in [11], we can get that for every $l\ge 1$

$$\|W_n^l\|_{L_{t,x}^3[I_n]}^3 = \|\sum_{j=1}^{l} U_n^j\|_{L_{t,x}^3[I_n]}^3 \to \sum_{j=1}^{l}\|U_n^j\|_{L_{t,x}^3[I_n]}^3, \quad \text{as} \quad n\to\infty.$$

Meanwhile by (4.3), the series $\sum\|V^j\|_{L^2}^2$ converge. Thus for every $\epsilon>0$, there exists $l(\epsilon)$ such that

$$\|V^j\|_{L^2}\le\epsilon, \quad ,\forall j>l(\epsilon).$$

The theory of small data asserts that , for ϵ sufficiently small, U^j is global and

$$\|U^j\|_{L_{t,x}^3} \lesssim \|V^j\|_{L^2},$$

which yields that

$$\sum_{j>l(\epsilon)} \|U^j\|^3_{L^3_{t,x}} < \infty.$$

So we have to deal only with a finite number of nonlinear profiles $\{U^j\}_{1\le j\le l(\epsilon)}$. But in view of the pairwise orthogonality of $\{\mathbf{\Gamma}^j_n\}_{j=1}^\infty$, one has

$$\lim_{n\to\infty} \Big\| \sum_{j=1}^{l(\epsilon)} U^j_n \Big\|_{L^3_{t,x}[I_n]} \le \sum_{j=1}^{l(\epsilon)} \lim_{n\to\infty} \Big\| U^j_n \Big\|_{L^3_{t,x}[I_n]} < \infty$$

and then (4.14) follows.

Now, we estimate (4.9).

$$\begin{aligned}
&\Big\| p(W^l_n + w^l_n) - p(W^l_n) \Big\|_{L^1_t L^2_x[I_n]} \\
\lesssim &\Big\| \big(|x|^{-2} * |W^l_n + w^l_n|^2\big) w^l_n \Big\|_{L^1_t L^2_x[I_n]} + \Big\| \big(|x|^{-2} * (W^l_n w^l_n)\big) w^l_n \Big\|_{L^1_t L^2_x[I_n]} + \Big\| \big(|x|^{-2} * |w^l_n|^2\big) W^l_n \Big\|_{L^1_t L^2_x[I_n]} \\
\lesssim &\Big\| W^l_n \Big\|^2_{L^3_{t,x}[I_n]} \|w^l_n\|_{L^3_{t,x}[I_n]} + \|w^l_n\|^2_{L^3_{t,x}[I_n]} \Big(\|W^l_n\|_{L^3_{t,x}[I_n]} + \|w^l_n\|_{L^3_{t,x}[I_n]} \Big) \\
= &o_n(1).
\end{aligned}$$

The last equality is due to (4.14) and the fact that $\|w^l_n\|_{L^3_{t,x}[I_n]} \to 0$ as $l \to \infty$.

(4.10) can be estimated similarly. In fact, we have

$$(4.10) \lesssim \Big(\|W^l_n + w^l_n\|^2_{L^3_{t,x}[I_n]} \|r^l_n\|_{L^3_{t,x}[I_n]} + \|W^l_n + w^l_n\|_{L^3_{t,x}[I_n]} \|r^l_n\|^2_{L^3_{t,x}[I_n]} + \|r^l_n\|^3_{L^3_{t,x}[I_n]} \Big).$$

Now we can prove (4.7). Collecting all the previous facts, we have

$$\begin{aligned}
&\sup_{t\in I_n} \|r^l_n\|_{L^2} + \|r^l_n\|_{L^3_{t,x}[I_n]} \\
\le &C \Big(\|W^l_n + w^l_n\|_{L^3_{t,x}[I_n]} \|r^l_n\|_{L^3_{t,x}[I_n]} + \|r^l_n\|^3_{L^3_{t,x}[I_n]} + \|r^l_n\|^2_{L^3_{t,x}[I_n]} + \|r^l_n(0,\cdot)\|_{L^2} \Big) + o_n(1).
\end{aligned} \tag{4.15}$$

As in [12], for every $\varepsilon > 0$ we can divide $I^+_n = I_n \cap \mathbb{R}_+$ into finite n-dependent intervals, namely,

$$I^+_n = [0, a^1_n] \cup [a^1_n, a^2_n] \cup \cdots \cup [a^{p-1}_n, a^p_n),$$

with each interval denoted by I^i_n $(i = 1, 2, \cdots, p)$, such that for every $1 \le i \le p$ and every $l \ge 1$,

$$\limsup_{n\to\infty} \|W^l_n + w^l_n\|_{L^3_{t,x}(I^i_n \times \mathbb{R}^4)} \le \varepsilon.$$

The $I^-_n = I_n \cap \mathbb{R}_-$ can be similarly dealt with. Applying (4.15) on I^1_n, it follows that

$$\sup_{t\in I^1_n} \|r^l_n\|_{L^2} + \|r^l_n\|_{L^3_{t,x}[I^1_n]} \lesssim \epsilon \|r^l_n\|_{L^3_{t,x}[I^1_n]} + \|r^l_n\|^3_{L^3_{t,x}[I^1_n]} + \|r^l_n\|^2_{L^3_{t,x}[I^1_n]} + \|r^l_n(0,\cdot)\|_{L^2} + o_n(1).$$

By choosing ϵ sufficiently small, we obtain

$$\sup_{t\in I^1_n} \|r^l_n\|_{L^2} + \|r^l_n\|_{L^3_{t,x}[I^1_n]} \lesssim \|r^l_n(0,\cdot)\|_{L^2} + \sum_{\alpha=2}^{3} \|r^l_n\|^\alpha_{L^3_{t,x}[I^1_n]} + o(1).$$

Observe that, by the definition of the nonlinear profile U^j_n, we have

$$\lim_{n\to\infty} \|r^l_n(0,\cdot)\|_{L^2} = 0$$

for every $l \geq 1$. This fact and a standard bootstrap argument show easily that

$$\lim_{n\to\infty} \Big(\sup_{t\in I_n^1} \|r_n^l\|_{L^2} + \|r_n^l\|_{L^3_{t,x}[I_n^1]} \Big) \overset{l\to\infty}{\longrightarrow} 0.$$

This gives, in particular

$$\lim_{n\to\infty} \|r_n^l(a_n^1,\cdot)\|_{L^2} \overset{l\to\infty}{\longrightarrow} 0$$

and allows us to repeat the same argument on I_n^2. We iterate the same process for every $1 \leq i \leq p$. Since $I = I_n^1 \cup I_n^2 \cup \cdots \cup I_n^p$ and p is finite independently of n and l, we get

$$\lim_{n\to\infty} \big(\|r_n^l\|_{L^3_{t,x}[I_n]} + \sup_{t\in I_n} \|r_n^l\|_{L^2}\big) \to 0$$

as $l \to \infty$, which is (4.7).

Step 2: Now we prove the equivalence of (i) and (ii).

$(i) \Rightarrow (ii)$:

Suppose that for all j, $\lim\limits_{n\to\infty} \|\Gamma_n^j U^j\|_{L^3_{t,x}[I_n]} < +\infty$, then

$$\big\|u_n\big\|_{L^3_{t,x}[I_n]} \leq \sum_{j=1}^{l} \big\|U_n^j\big\|_{L^3_{t,x}[I_n]} + \big\|r_n^l\big\|_{L^3_{t,x}[I_n]} + \big\|w_n^l\big\|_{L^3_{t,x}[I_n]}.$$

From (4.2), we have

$$\limsup_{n\to\infty} \|w_n^l\|_{L^3_{t,x}[I_n]} \xrightarrow{l\to\infty} 0 \quad \text{and} \quad \lim_{n\to\infty} \|r_n^l\|_{L^3_{t,x}[I_n]} \xrightarrow{l\to\infty} 0.$$

It immediately follows that

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}[I_n]} < +\infty.$$

$(ii) \Rightarrow (i)$:

If (i) does not hold, there exists a family of $\tilde{I}_n \subset I_n$ with 0 included, such that

$$\sum_{j=1}^{\infty} \lim_{n\to\infty} \big\|U_n^j\big\|^3_{L^3_{t,x}[\tilde{I}_n]} > M$$

for arbitrary large M and

$$\|u_n\|_{L^3_{t,x}[\tilde{I}_n]} < \infty.$$

By the orthogonality, we have

$$\lim_{n\to\infty} \|u_n\|^3_{L^3_{t,x}[\tilde{I}_n]} \geq \sum_{j=1}^{\infty} \lim_{n\to\infty} \|U_n^j\|^3_{L^3_{t,x}[\tilde{I}_n]} > M.$$

This leads to

$$\lim_{n\to\infty} \|u_n\|^3_{L^3_{t,x}[I_n]} \geq \lim_{n\to\infty} \|u_n\|^3_{L^3_{t,x}[\tilde{I}_n]} > M,$$

which implies that

$$\lim_{n\to\infty} \|u_n\|_{L^3_{t,x}[I_n]} = +\infty.$$

This contradicts (ii). This completes the proof of Theorem 4.1.

Proof of Theorem 1.3. We choose $\{u_{0,n}\}$ such that $\|u_{0,n}\|_{L^2} \downarrow \delta_0$, let u_n is the solution of (1.3) with data $u_{0,n}$. By the definition of δ_0, we can assume that the interval of existence for u_n is finite. By time translation and scaling, we may assume that $\{u_n\}_{n=1}^{\infty}$ is well defined on $[0,1]$, and

$$\lim_{n\to\infty} \|u_n\|_{L_t^3([0,1],L_x^3)} = +\infty.$$

Let $\{U^j, V^j, \rho_n^j, s_n^j, \xi_n^j, x_n^j\}$ be the family of linear and nonlinear profiles associated to $\{u_n\}_{n=1}^{\infty}$ via Lemma 4.1 and Theorem 4.1. Then the equivalence in Theorem 4.1 implies that there exists a j_0 such that U^{j_0} blows up. On one hand, by the definition of B_{δ_0},

$$\|V^{j_0}\|_{L^2} \geq \delta_0.$$

On the other hand, we have

$$\sum_{j\geq 0} \|V^{j_0}\|_{L^2}^2 \leq \lim_{n\to\infty} \|u_{0,n}\|_{L^2}^2 = \delta_0^2.$$

Thus by mass conservation and the definition of nonlinear profile, we have

$$\|U^{j_0}\|_{L^2} = \|V^{j_0}\|_{L^2} \leq \delta_0.$$

Therefore,

$$\|U^{j_0}\|_{L^2} = \delta_0.$$

Because U^{j_0} is the solution of (1.3) satisfying $U(s^{j_0}, x) = V(s^{j_0}, x)$, where $s^{j_0} = \lim_{n\to\infty} s_n^{j_0}$. If s^{j_0} is finite, then U^{j_0} is the blow up solution with minimal mass. If $s^{j_0} = \infty$, we can use the pseudo-conformal transformation to get a blow up solution with minimal mass. This shows the existence of initial data such that solution of (1.3) blows up in finite time for $t > 0$. In the proof of Theorem 1.4 we will show that there exists an initial data $u_0 \in L^2(\mathbb{R}^4)$ with $\|u_0\|_{L^2} = \delta_0$, such that the solution u of (1.3) blows up for both $t > 0$ and $t < 0$.

Proof of Theorem 1.4. (i) Suppose u is a solutions of (1.3) which blows up at finite time $T^* > 0$ and $\{t_n\}_{n=1}^{\infty}$ is a sequence increasingly going to T^* as $n \to \infty$. Let

$$u_n(t,x) = u(t_n + t, x),$$

then $\{u_n\}_{n=1}^{\infty}$ is a family of solutions on $I_n = [-t_n, T^* - t_n)$. Moreover, we have

$$\lim_{n\to\infty} \Big\| u_n \Big\|_{L^3_{t,x}\in[0,T^*-t_n)} = \lim_{n\to\infty} \Big\| u_n \Big\|_{L^3_{t,x}\in[-t_n,0]} = +\infty.$$

Since $\|u_n\|_{L^2}$ is bounded due to L^2 conservation, we can apply Lemma 4.1 and then Theorem 4.1 on $I_n = [0, T^* - t_n)$ to get that there exists some j_0 such that the nonlinear profile $\{U^{j_0}, \rho_n^{j_0}, s_n^{j_0}, \xi_n^{j_0}, x_n^{j_0}\}$ satisfies

$$\lim_{n\to\infty} \Big\| U^{j_0} \Big\|_{L^3_{t,x}[I_n^{j_0}]} = +\infty, \tag{4.16}$$

where

$$I_n^{j_0} := [s_n^{j_0}, (\rho_n^{j_0})^2 (T^* - t_n) + s_n^{j_0}).$$

In fact, let $s^{j_0} = \lim_{n\to\infty} s_n^{j_0}$, then $s^{j_0} \neq \infty$, otherwise, $I_n^{j_0} \to \emptyset$ and (4.16) is impossible. This implies either $s^{j_0} = -\infty$ or $s^{j_0} = 0$ (up to translation). If $s^{j_0} = 0$, let U^{j_0} be the solution of (1.4) with initial data V^{j_0}, then (4.16) implies U^{j_0} blows up at time $T^*_{j_0} \in (0, +\infty)$ and

$$\lim_{n\to\infty} (\rho_n^{j_0})^2 (T^* - t_n) \geq T^*_{j_0}. \tag{4.17}$$

If we assume also that $\|u_0\|_{L^2} < \sqrt{2}\delta_0$, then there is at most one linear profile with L^2-norm greater than δ_0 thanks to (4.3). That means that the profile U^{j_0} founded above is the only blow up nonlinear profile (since all the other profiles have L^2 norm less than δ_0 and then they are global). By repeating the same argument in $I_n = [-t_n, 0]$, we get

$$\lim_{n\to\infty} \|U^{j_0}\|_{L^3_{t,x}[I_n^{j_0}]} = +\infty, \quad I_n^{j_0} = [-(\rho_n^{j_0})^2 t_n + s_n^{j_0}, s_n^{j_0}].$$

This implies that $s^{j_0} \neq -\infty$. Hence $s^{j_0} = 0$ and the solution U^{j_0} of (1.3) with initial data $V^{j_0}(0,\cdot)$ blows up also for $t < 0$. Thus the nonlinear profile U^{j_0} is the solution of (1.3) which blows up for both $t < 0$ and $t > 0$.

(ii) The linear decomposition yields

$$(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n,\cdot)) = V^{j_0} + \sum_{1\le j\le l; j\ne j_0} (\mathbf{\Gamma}_n^{j_0})^{-1}\mathbf{\Gamma}_n^j V^j + (\mathbf{\Gamma}_n^{j_0})^{-1} w_n^l.$$

The family $\{\mathbf{\Gamma}_n^j\}_{j=1}^\infty$ is pairwise orthogonal, so for every $j \neq j_0$,

$$(\mathbf{\Gamma}_n^{j_0})^{-1}\mathbf{\Gamma}_n^j V^j \xrightarrow{n\to\infty} 0 \text{ weakly in } L^2.$$

Then

$$(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n,\cdot)) \xrightarrow{n\to\infty} V^{j_0} + \tilde{w}^l \quad \text{weakly.}$$

where $\tilde{w}^l$ denote the weak limit of $(\mathbf{\Gamma}_n^{j_0})^{-1} w_n^l$. However, we have

$$\|\tilde{w}^l\|_{L^3_{t,x}} \le \lim_{n\to\infty} \|w_n^l\|_{L^3_{t,x}} \xrightarrow{l\to+\infty} 0.$$

By the uniqueness of weak limit, we get $\tilde{w}^l = 0$ for every $l \geq j_0$. Hence, we obtain

$$(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n,\cdot)) \xrightarrow{n\to\infty} V^{j_0}.$$

We need the following lemma:

Lemma 4.2 ([21]). *Let $\{\varphi_n\}_{n\ge1}$ and φ be in $L^2(\mathbb{R}^4)$. The following statement is equivalent:*

(1) $\varphi_n \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R}^4)$.

(2) $e^{it\Delta}\varphi_n \rightharpoonup e^{it\Delta}\varphi$ in $L^3_{t,x}(\mathbb{R}^{4+1})$

Applying this lemma to $(\mathbf{\Gamma}_n^{j_0})^{-1}(e^{it\Delta}(u(t_n,\cdot))$, we get

$$e^{-is_n\Delta}\Big(\rho_n^2 e^{ix\cdot\xi_n} e^{i\theta_n} u(t_n, \rho_n x + x_n)\Big) \rightharpoonup V^{j_0}(0,\cdot)$$

with

$$s_n = s_n^{j_0}, \quad \rho_n = \frac{1}{\rho_n^{j_0}}, \quad \theta_n = \frac{x_n^{j_0}\xi_n^{j_0}}{\rho_n^{j_0}}, \quad x_n = \frac{-x_n^{j_0}}{\rho_n^{j_0}}, \quad \xi_n = -\frac{\xi_n^{j_0}}{\rho_n^{j_0}}.$$

Up to subsequence, we can assume that $e^{i\theta_n} \to e^{i\theta}$. Since $s_n \to 0$, we get

$$\rho_n^2 e^{ix\cdot\xi_n} u(t_n, \rho_n x + x_n) \rightharpoonup e^{-i\theta} V^{j_0}(0,\cdot). \tag{4.18}$$

The associated solution is $e^{-i\theta}U^{j_0}$. (4.17) gives

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \le \frac{1}{\sqrt{T^*_{j_0}}}.$$

This completes the proof of Theorem 1.4.

(iii) Let u be a solution of (1.1) with $\|u_0\|_{L^2} < \sqrt{2}\delta_0$, which blows up at finite time $T^* > 0$. Let $\{t_n\}_{n=1}^{\infty}$ be any time sequence such that $t_n \uparrow T^*$ as $n \to \infty$. So there exist $V \in L^2(\mathbb{R}^4)$ with $\|V\|_{L^2} \geq \delta_0$ and a sequence $\{\rho_n, \xi_n, x_n\} \subset \mathbb{R}_+^* \times \mathbb{R}^4 \times \mathbb{R}^4$ such that up to a subsequence,

$$(\rho_n)^2 e^{ix\cdot\xi_n} u(t_n, \rho_n x + x_n) \xrightarrow{n\to\infty} V$$

and

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T^* - t_n}} \leq A$$

for some $A \geq 0$. Thus we have

$$\lim_{n\to\infty} \rho_n^4 \int_{|x|\leq R} |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x|\leq R} |V|^2 dx$$

for every $R \geq 0$. This implies that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq R\rho_n} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |V|^2 dx.$$

Since $\frac{\sqrt{T^*-t}}{\lambda(t)} \to 0$ as $t \uparrow T^*$, it follows that $\frac{\rho_n}{\lambda(t_n)} \to 0$ and then

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t_n)} |u(t_n, x)|^2 dx \geq \int |V|^2 dx \geq \delta_0^2.$$

Since $\{t_n\}_{n=1}^{\infty}$ is an arbitrary sequence, we infer

$$\liminf_{t\to T} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t)} |u(t, x)|^2 dx \geq \delta_0^2.$$

However for every $t \in [0, T)$, the function $y \mapsto \int_{|x-y|\leq\lambda(t)} |u(t, x)|^2 dx$ is continuous and goes to 0 at infinity. As a consequence, we get

$$\sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\lambda(t)} |u(t, x)|^2 dx = \int_{|x-x(t)|\leq\lambda(t)} |u(t, x)|^2 dx$$

for some $x(t) \in \mathbb{R}^4$ and this completes the proof of Theorem 1.4.

Proof of Corrolary 1.2. In context of the proof of Theorem 1.4 we assume also that

$$\|u_n\|_{L^2} = \|u_0\|_{L^2} = \delta_0.$$

(4.3) gives that

$$\|V^{j_0}\|_{L^2} \leq \delta_0.$$

It follows that

$$\|V^{j_0}\|_{L^2} = \delta_0.$$

This implies that there exists a unique profile V^{j_0} and the weak limit in (4.18) is strong.

Acknowledgement: The authors thank professor S. Keraani for several lectures on his work [9] and helpful discussions, and thank Pin Yu for helpful discussion. C. Miao and G. Xu were partly supported by the NSF of China (No.10725102, No.10726053), and L. Zhao was supported by China postdoctoral science foundation project.

References

[1] V. Banica, Remarks on the blow-up for the Schrödinger equation with critical mass on a plane domain. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3 (2004), no. 1, 139-170.

[2] P. Bégout and A. Vargas, Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equaiton. Trans. Amer. Math. Soc., 359(2007), 5257-5282.

[3] J. Bourgain, Refinements of Strichartz inequaltiy and applications to 2D-NLS with critical nonlinearity. IMRN, 8(1998) 253-283.

[4] R. Carles and S. Keraani, On the role of quadratic oscillations in nonlinear Schrödinger equation II. The L^2-critical case. Trans. Amer. Math. Soc., 359(2007), 33-62.

[5] T. Cazenave, *Semilinear Schrödinger equations*. Courant Lecture Notes in Mathematics, Vol. 10. New York: New York University Courant Institute of Mathematical Sciences, 2003.

[6] J. Fröhlich and E. Lenzmann, Mean-field limit of quantum Bose gases and nonlinear Hartree equation. Séminaire: Équations aux Dérivées Partielles. 2003-2004, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 2004, pp. Exp. No. XIX, 26.

[7] J. Ginibre, *Introduction aux équations de Schrödinger non linéaires.* Master course, 94-95.

[8] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of Hartree equations. Nonlinear wave equations (Providence, RI, 1998), 29-60, Contemp. Math., 263, Amer. Math. Soc., Providence, RI, 2000.

[9] T. Hmidi and S. Keraani, Blowup theory for the critical nonlinear Schrödinger equations revisited. IMRN, 46(2005) , 2815-2828.

[10] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math., 120:5(1998), 955-980.

[11] S. Keraani, On the defect of compactness for the Strichartz estimates of the Schrödinger equations. J. Diff. Equa., 175(2001) 353-392.

[12] S. Keraani, On the blow up phenomenon of the critical nonlinear Schrödinger equation. J. Funct. Anal., 235(2006), 171-192.

[13] J. Krieger, E. Lenzmann and P. Raphael, On stability of pseudo-conformal blowup for L^2-critical Hartree equation. arXiv:0808.2324.

[14] M. K. Kwong, Uniequeness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$. Arch. Rat. Mech. Anal., 105(1989), 243-266.

[15] D. Li, C. Miao and X. Zhang, The focusing energy-critical Hartree equation. To appear in J. Diff. Equa..

[16] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquar's nonlinear equation. Stud. Appl. Math., 57 (1977), 93-105.

[17] S. Liu, Regularity, symmetry, and uniqueness of some integral type quasilinear equations with nonlocal nonlinearities. Preprint.

[18] F. Merle, Blow-up phenomena for critical nonlinear Schrödinger and Zakharov equations. Proceeding of the International Congress of Mathematicians (Berlin, 1998), Doc. Math. extra. Vol. III(1998), 57-66.

[19] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equaitons with critical power. Duke Math. J., 69:2(1993), 427-454.

[20] F. Merle and Y. Tsutsumi, L^2 concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity. J. Diff. Equa., 84(1990), 205-214.

[21] F. Merle and L. Vega, Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D. IMRN, 8(1998), 399-425.

[22] F. Merle and P. Raphael, Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation. GAFA, 13(2003), 591-642.

[23] F. Merle and P. Raphael, On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation. Invent. Math., 156 (2004), 565-672.

[24] F. Merle and P. Raphael, On a sharp lower bound on the blow-up rate for the L^2 critical nonlinear Schrödinger equation. J. Amer. Math. Soc., 19:1(2005), 37-90.

[25] C. Miao, G. Xu and L. Zhao, The Cauchy problem of the Hartree equation. J. PDE, 21(2008), 22-44.

[26] C. Miao, G. Xu and L. Zhao, Global well-posedness and scattering for the energy-critical, defocusing Hartree equation for radial data. J. Funct. Anal., 253(2007), 605-627.

[27] C. Miao, G. Xu and L. Zhao, Global well-posedness and scattering for the energy-critical, defocusing Hartree equation in $\mathbb{R}^{1+n}$. Preprint.

[28] C. Miao, G. Xu and L. Zhao, Global well-posedness, scattering and blow-up for the energy-critical, focusing Hartree equation in the radial case. To appear in Coll. Math.

[29] C. Miao, G. Xu and L. Zhao, Global well-posedness and scattering for the mass-critical Hartree equation with radial data. To appear in J. Math. Pures Appl..

[30] K. Nakanishi, Energy scattering for Hartree equations. Math. Res. Lett., 6(1999), 107-118.

[31] H. Nawa, "Mass concentration" phenomenon for the nonlinear Schrödinger equation with the critical power nonlinearity. Fukcial. Ekvac., 35:1(1992), 1-18.

[32] T. Tao, M. Visan, and X. Zhang, Minimal-mass blowup solutions of the mass-critical NLS. To appear in Forum Math.

[33] R. Killip, T. Tao and M. Visan, The cubic nonlinear Schrödinger equation in two dimensions with radial data. Preprint.

[34] R. Killip, M. Visan and X. Zhang, The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher. Preprint.

[35] M. Weinstein, The Nonlinear Schrödinger Equation-Singularity Formation, Stability and Dispersion. pp. 213-232 in: The connection between infinite-dimensional and finite-dimensional dynamical systems, Contemporary Math., No. 99, Amer. Math. Soc., Providence, R. I., 1989.

[36] http://tosio.math.toronto.edu/wiki/index.php/Hartree equation.